On Keller Theorem for Anisotropic Media

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Abstract

The Keller theorem in the problem of effective conductivity in anisotropic two-dimensional (2D) many-component composites makes it possible to establish a simple inequality $\sigma^e_{\mathsf{is}}(\sigma_i^{-1})\cdot\sigma^e_{\mathsf{is}}(\sigma_k)>1$ for the isotropic part $\sigma^e_{\mathsf{is}}(\sigma_k)$ of the 2-nd rank symmetric tensor $\widehat{\sigma}^e_{i,j}$ of effective conductivity.

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The extension of the Keller theorem [1] in the problem of effective conductivity in the infinite 2D two-component composites on the many-component case [2] poses a new question on the duality relation for the 2-nd rank symmetric tensor $\widehat{\sigma}^e_{i,j}$ of effective conductivity in anisotropic media. It is related to the restrictions imposed on the linear invariant of $\widehat{\sigma}^e_{i,j}$ which is called an isotropic part $\sigma^e_{\mathsf{is}}(\sigma_k)$ of effective conductivity. Recently the perturbation theory for the infinite periodic three-component 2D checkerboard with two-fold rotation lattice symmetry was developed [3] where the coincidence of $\sigma^e_{\mathsf{is}}(\sigma_k)$ with solution $\sigma_B(\sigma_k)$ of Bruggemann Eqn was established up to the 6-th order term. This fact is curious because it gives grounds to think that Effective Medium Approximation (EMA) describes exactly $\sigma^e_{\mathsf{is}}(\sigma_k)$ in this certain structure. Here we will discuss this conclusion.

Let us define the isotropic part of conductivity tensor

$$\sigma^e_{\mathsf{is}}(\sigma_k)=\frac{1}{2}\,\mathsf{Tr}\;\widehat{\sigma}^e_{i,j}(\sigma_k)\;,\quad k=1,2,...,n\;, \tag{1}$$

which is an invariant scalar with respect to the plane rotation and recall the Keller theorem for the principal values $\widehat{\sigma}^{\mathsf{xx}}_e$, $\widehat{\sigma}^{\mathsf{yy}}_e$ of diagonalized matrix $\widehat{\sigma}^{ij}_e$ for 2D n-component composite

$$\begin{aligned}&\widehat{\sigma}^{\mathsf{xx}}_e(\sigma_1^{-1},\sigma_2^{-1},...,\sigma_n^{-1})\cdot\widehat{\sigma}^{\mathsf{yy}}_e(\sigma_1,\sigma_2,...,\sigma_n)=1\;,\\&\widehat{\sigma}^{\mathsf{yy}}_e(\sigma_1^{-1},\sigma_2^{-1},...,\sigma_n^{-1})\cdot\widehat{\sigma}^{\mathsf{xx}}_e(\sigma_1,\sigma_2,...,\sigma_n)=1\;.\end{aligned} \tag{2}$$

Both (1) and (2) make us possible to derive a simple inequality for $\Lambda^e_{\mathsf{is}}=\sigma^e_{\mathsf{is}}(\sigma_i^{-1})\cdot\sigma^e_{\mathsf{is}}(\sigma_k)$

$$\begin{aligned}\Lambda^e_{\mathsf{is}}=&\frac{1}{4}\left[2+\widehat{\sigma}^{\mathsf{xx}}_e(\sigma_k)\cdot\widehat{\sigma}^{\mathsf{xx}}_e(\sigma_k^{-1})+\widehat{\sigma}^{yy}_e(\sigma_k)\cdot\widehat{\sigma}^{\mathsf{yy}}_e(\sigma_k^{-1})\right]=\\&\frac{1}{4}\left[2+\frac{\widehat{\sigma}^{\mathsf{xx}}_e(\sigma_k)}{\widehat{\sigma}^{\mathsf{yy}}_e(\sigma_k)}+\frac{\widehat{\sigma}^{\mathsf{yy}}_e(\sigma_k)}{\widehat{\sigma}^{\mathsf{xx}}_e(\sigma_k)}\right]\geq 1\;,\end{aligned} \tag{3}$$

where the only isotropic media $\widehat{\sigma}^{\mathsf{xx}}_e=\widehat{\sigma}^{\mathsf{yy}}_e$ corresponds to the equality in (3). At the same time another isotropic invariant $\Delta^e_{\mathsf{is}}=\det\widehat{\sigma}^e_{ij}(\sigma_k)$ satisfies the duality relation

$$\Delta^e_{\mathsf{is}}(\sigma_k)\cdot\Delta^e_{\mathsf{is}}(\sigma_k^{-1})=1\;.$$

The EMA theory of the infinite 2D n-component isotropic comosite has its consequence the Bruggemann Eqn [4]

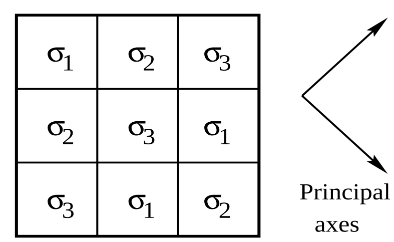
$$\sum_{k=1}^{n}\frac{\sigma_k-\sigma_B(\sigma_k)}{\sigma_k+\sigma_B(\sigma_k)}=0\;, \tag{4}$$

which necessarely leads to the duality relation

$$\sigma_B(\sigma_k^{-1})\cdot\sigma_B(\sigma_k)=1 \tag{5}$$

that reflects both the conformal invariance of the Maxwell Eqns in 2D isotropic comosite and S_n-permutation invariance of the n-colour tessellation of the plane. The latter means that σ^e_{is} can satisfy the Bruggemann Eqn only for isotropic S_n-permutation invariant media: $\widehat{\sigma}^{xx}_e=\widehat{\sigma}^{yy}_e$, $\widehat{\sigma}^{xy}_e=0$ in any reference frame $\{x,y\}$.

The infinite periodic 2D three-component checkerboard was considered in [3] for symmetrically related partial conductivities ($\sigma_1=1,\sigma_{2,3}=1\pm\delta$).



Such structure doesn't possess an isotropy of the 2-nd rank conductivity tensor $\widehat{\sigma}^{i,j}_e$ that follows from the simple crystallographycal consideration [5] as well as from the straightforward calculation [3] of the non-diagonal term $\widehat{\sigma}^{xy}_e\propto\delta^2$. Therefore $\sigma^e_{\mathsf{is}}(\sigma_k)$ for this structure can not satisfy the Bruggemann Eqn (4) even if its coincidence with $\sigma_B(\sigma_k)$ riched the δ^6 term in the perturbation theory.

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